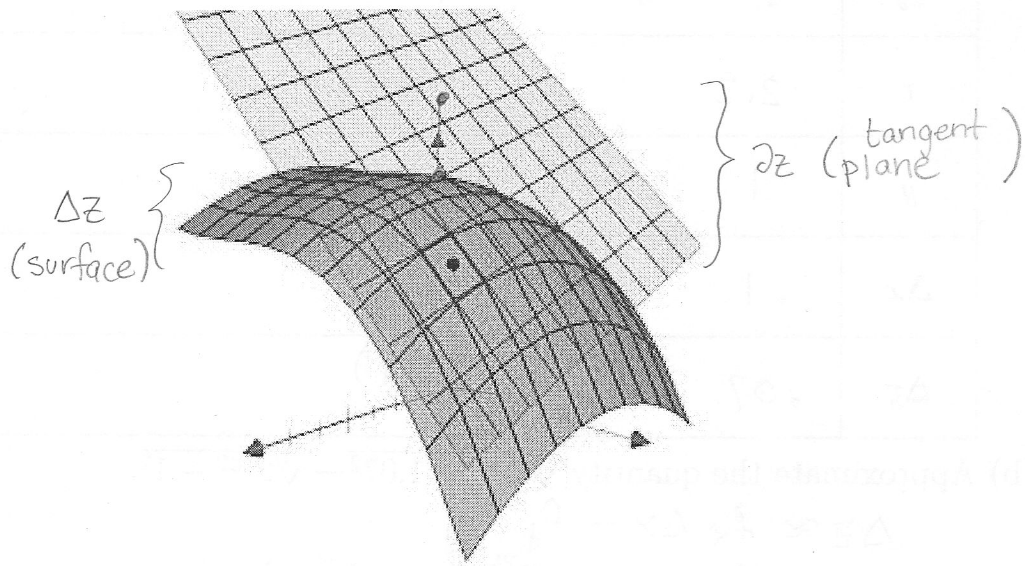
Lesson 21

If $z = f(x, y)$, then we have differentials ∂z, ∂x, and ∂y, which are related by the following formula: $\partial z = f_x(x, y)\partial x + f_y(x, y)\partial y$ (this is called the *total differential).*

We can approximate Δz by $\Delta z \approx f_x(x, y)\Delta x + f_y(x, y)\Delta y$.



1. Find ∂z for $z = e^{x^2+y^2}\tan(2x)$.

$$\frac{\partial z}{\partial x} = 2xe^{x^2+y^2}\tan(2x) + e^{x^2+y^2}(2\sec^2(2x))$$

$$\frac{\partial z}{\partial y} = 2ye^{x^2+y^2}\tan(2x)$$

$$\partial z = \left[2xe^{x^2+y^2}\tan(2x) + e^{x^2+y^2}(2\sec^2(2x))\right]\partial x + 2ye^{x^2+y^2}\tan(2x)\,\partial y$$

2. Use the total differential to approximate the quantity $\sqrt[3]{2.6^2 - 1.07^2} - \sqrt[3]{2.5^2 - 1^2}$ to 3 decimal places.

$$\| $$
$$f(2.6, 1.07) - f(2.5, 1)$$

(a) Fill in the table:

$f(x, y)$	$\sqrt[3]{x^2-y^2} = (x^2-y^2)^{1/3}$	
f_x	$\frac{1}{3}(x^2-y^2)^{-2/3}(2x)$	
f_y	$\frac{1}{3}(x^2-y^2)^{-2/3}(-2y)$	
x	2.5	
y	1	
Δx	.1	$(2.6 - 2.5)$
Δy	.07	$(1.07 - 1)$

(b) Approximate the quantity $\sqrt[3]{2.6^2 - 1.07^2} - \sqrt[3]{2.5^2 - 1^2}$.

$$\Delta z \approx f_x \, \Delta x + f_y \, \Delta y$$
$$= \frac{1}{3}(2.5^2 - 1^2)^{-2/3}(2)(2.5)(.1) + \frac{1}{3}(2.5^2 - 1^2)^{-2/3}(-2)(1)(.07)$$
$$\approx .0397$$
$$\rightarrow \boxed{.040}$$

actual: .0393

3. A company's profit is given by $P(K, L) = 500K^{1/3}L^{2/3}$, where K is the company's overhead costs in thousands and L is the number of workers in hundreds. $\Delta P \leftarrow$ Find the change in profit when the overhead costs are currently 3 million dollars and there are 2,500 workers and overhead costs are decreased by 5 thousand dollars while the number of workers is increased by 150.

(a) Fill in the table:

$P(K, L)$	$500K^{1/3}L^{2/3}$
$P_K = \frac{\partial P}{\partial K}$	$500\left(\frac{1}{3}K^{-2/3}\right)L^{2/3}$
$P_L = \frac{\partial P}{\partial L}$	$500K^{1/3}\left(\frac{2}{3}L^{-1/3}\right)$
K	3,000,000 = 3,000 thousands
L	2,500 = 25 hundred
ΔK	− 5 thousand
ΔL	1.5 hundred

(b) Approximate ΔP.

$$\Delta P \approx P_K \, \Delta K + P_L \, \Delta L$$
$$= 500\left(\tfrac{1}{3}(3000)^{-2/3}\right)(25)^{2/3}(-5)$$
$$+ 500(3000)^{1/3}\left(\tfrac{2}{3}(25)^{-2/3}\right)(1.5)$$
$$= \$24\,31.96$$

4. Recall that $A = Pe^{rt}$. Suppose you deposit \$1,000 today into a 5 year CD and interest is compounded continuously at an annual rate of 2%.

(a) How much money will be in the CD after 5 years?

$$A = 1000e^{.02(5)} \approx \$1,105.17$$

(b) Suppose the rate changes to 1.95%. Fill in the following table:

$A(P, r)$	$Pe^{r(5)} = Pe^{5r}$
$A_P = \frac{\partial A}{\partial P}$	e^{5r}
$A_r = \frac{\partial A}{\partial r}$	$5Pe^{5r}$
P	1000
r	.02
ΔP	?
Δr	$-.0005$ $(.0195 - .02)$

(c) Approximately how much more will you need to deposit today to obtain the same amount in 5 years?

$$\Delta A \approx A_P \Delta P + A_r \Delta r$$

$$0 = e^{5(.02)} \Delta P + 5(1000)e^{5(.02)}(-.0005)$$

$$\Delta P = \$2.50$$

5. A tank is a cylinder h feet tall with radius r feet. Recall the surface area of a cylinder with no top is $A(r,h) = \pi r2 + 2\pi rh$. A particular tank is measured to be 6 feet tall with a radius of 3 feet. The height is measured with an error of at most 3 inches (1/4 of a foot) and the radius is measured with a maximum error of 1 inch (1/12 of a foot).

(a) Fill in the following table:

$A(h,r)$	$\pi r^2 + 2\pi rh$
$A_h = \frac{\partial A}{\partial h}$	$2\pi r$
$A_r = \frac{\partial A}{\partial r}$	$2\pi r + 2\pi h$
h	6
r	3
Δh	$\pm \frac{1}{4}$
Δr	$\pm \frac{1}{12}$

(b) What is the maximum error in the calculation of the surface area?

$$\Delta A \approx A_h \Delta h + A_r \Delta r$$
$$= 2\pi(3)\Delta h + [2\pi(3) + 2\pi(6)]\Delta r$$

Δh	Δr	ΔA
1/4	1/12	9.425 $\leftarrow$ max.
1/4	-1/12	0
-1/4	1/12	0
-1/4	-1/12	-9.425

(c) What is the relative percentage error in calculating A?

max error $\rightarrow \frac{\Delta A}{A} \times 100\% = \frac{9.425}{\pi(3)^2 + 2\pi(3)(6)} \times 100\% = \boxed{6.67\%}$